Black body radiation &
Einstein and Debye theories of specific heats of solids.

e-content for B.Sc Physics (Honours)

B.Sc Part-I Paper-II

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Black body

1. A black body is an idealized physical body that absorbs all incident electromagnetic radiation, regardless of frequency or angle of incidence.
2. A black body in thermal equilibrium (that is, at a constant temperature) emits electromagnetic radiation called black-body radiation.
3. The radiation has a spectrum that is determined by the temperature alone, not by the body's shape or composition.
4. It is extremely difficult to realize a perfect black body, for which, the absorption of radiation is 100%. Transmission and reflection are zero.

To study blackbody radiation we need to revise some concept. They are given below

Thermodynamic equilibrium

1. In thermodynamic equilibrium all kinds of equilibrium hold at once.
2. It is characterized by no net macroscopic flows of matter or of energy.
3. Any microscopic exchanges are perfectly balanced.
4. The temperature is spatially uniform.
5. Entropy maximizes with equilibrium.

Thermodynamic state

- A thermodynamic system is a macroscopic object, the microscopic details of which are not explicitly considered in its thermodynamic description.

Internal energy

- It excludes the kinetic energy of motion of the system as a whole and the potential energy of the system as a whole due to external force fields.

Boltzmann constant

- The Boltzmann constant (kB or k), which is named after Ludwig Boltzmann, is a physical constant relating the average kinetic energy of particles in a gas with the temperature of the gas. It is the gas constant R divided by the Avogadro constant NA.

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Equipartition theorem

1. It relates the temperature of a system to its average energies in thermal equilibrium.
2. It assumes that energy is shared equally among all of its various modes. For example, the average kinetic energy per degree of freedom in translational motion of a molecule should equal that in rotational motion.
3. It gives the average values of individual components of the energy, such as, the kinetic energy of a particular particle, or the potential energy of a single spring. For example, it predicts that every atom in a monatomic ideal gas has an average kinetic energy of (3/2) kBT in thermal equilibrium.
4. When the thermal energy kBT is smaller than the quantum energy spacing in a particular degree of freedom (such as at lower temperatures), the average energy and heat capacity of this degree of freedom are less than the values predicted by equipartition.

5. Such decreases in heat capacity were among the first signs to physicists of the 19th century that classical physics was incorrect and that a new, more subtle, scientific model was required.
6. Along with other evidence, equipartition's failure to model black-body radiation—also known as the ultraviolet catastrophe—led Max Planck to suggest that energy in the oscillators in an object, which emit light, were quantized, a revolutionary hypothesis that spurred the development of quantum mechanics and quantum field theory.

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Rayleigh–Jeans law

1. The Rayleigh–Jeans law revealed an important error in physics theory of the time.
2. The law predicted an energy output that diverges towards infinity as wavelength approaches zero (as frequency tends to infinity).
3. Measurements of the spectral emission of actual black bodies revealed that the emission agreed with the Rayleigh–Jeans law at low frequencies but diverged at high frequencies; reaching a maximum and then falling with frequency, so the total energy emitted is finite.

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Ultraviolet catastrophe

1. The ultraviolet catastrophe was the prediction of classical physics that an ideal black body at thermal equilibrium will emit more energy as the frequency increases.
2. A blackbody would release an infinite amount of energy, contradicting the principles of conservation of energy.
3. The ultraviolet catastrophe results from the equipartition theorem of classical statistical mechanics which states that all harmonic oscillator modes (degrees of freedom) of a system at equilibrium have an average energy of (1/2)kT. It assumes that vibrating modes can increase infinitely.

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Black-body radiation

1. Black-body radiation is the thermal electromagnetic radiation within or surrounding a body.
2. It has a specific spectrum and intensity that depends only on the body's temperature.
3. As its temperature increases the peak of the spectrum shifts from infra-red toward higher frequencies of visible light.
4. Black-body radiation has a characteristic, continuous frequency spectrum.

5. If each Fourier mode of the equilibrium radiation in an otherwise empty cavity with perfectly reflective walls is considered as a degree of freedom capable of exchanging energy, then, according to the equipartition theorem of classical physics, there would be an equal amount of energy in each mode.
6. Since there are an infinite number of modes this implies infinite heat capacity (infinite energy at any non-zero temperature), as well as an unphysical spectrum of emitted radiation that grows without bound with increasing frequency, a problem known as the ultraviolet catastrophe.
7. Instead, in quantum theory the occupation numbers of the modes are quantized, cutting off the spectrum at high frequency in agreement with experimental observation and resolving the catastrophe. The study of the laws of black bodies and the failure of classical physics to describe them helped establish the foundations of quantum mechanics.

Explanation

1. The radiation from matter represents a conversion of a body's thermal energy into electromagnetic energy. At thermal equilibrium, matter emits and absorbs electromagnetic radiation. The electromagnetic radiation has a characteristic frequency distribution that depends on the temperature only.
2. At thermodynamic equilibrium the amount of every wavelength in every direction of thermal radiation emitted by a body at temperature T is equal to the corresponding amount that the body absorbs because it is surrounded by light at temperature T.
3. The black-body curve is characteristic of thermal light, which depends only on the temperature of the body. The principle of strict equality of emission and absorption is always upheld in a condition of thermodynamic equilibrium.
4. By making changes to Wien's radiation law consistent with thermodynamics and electromagnetism, Planck found a mathematical expression fitting the experimental data satisfactorily. Planck had to assume that the energy of the oscillators in the cavity was quantized, i.e., it existed in integer multiples of some quantity.
5. Einstein built on this idea and proposed the quantization of electromagnetic radiation itself in 1905 to explain the photoelectric effect.
6. These theoretical advances eventually resulted in the superseding of classical electromagnetism by quantum electrodynamics. These quanta were called photons and the black-body cavity was thought of as containing a gas of photons.
7. In addition, it led to the development of quantum probability distributions, called Fermi–Dirac statistics and Bose–Einstein statistics, each applicable to a different class of particles, fermions and bosons.

1 Introduction

At the end of 19th century scientists felt that all the laws of physics (which were known at that time) are enough to explain all the events occur in nature. It was believed that there are only two kinds in nature. First the bodies which are made up of particles and second radiation. All particles obey newton's laws of motion and radiation obey Maxwell's equations of electromagnetism. These laws (known at that time) are now a days known as classical physics.

Fortunately at the same time some experiments were performed. The results of some of the experiments could not be explained by the laws of so called classical physics. Some of them are Blackbody radiation, Photo electric effect and Compton effect etc. In order to explain these effects we need some new types of laws of physics which are known as Quantum physics. so Blackbody radiation is one of the great effects which leads us to modern physics. In this article we will see what is the blackbody radiation,how classical physics fails to explain its characteristics and how quantum principles takes birth. Let us start with simple definitions.

2 Some definitions

2.1 Radiation

According to Maxwell when a charge is accelerated it creates some electric and magnetic fields which are correlated to each other. They can move in form of a wave on space with finite velocity . These are known as electromagnetic radiation the velocity of the radiations is given by

$$v = \frac{1}{(\mu\epsilon)^{1/2}} \tag{1}$$

In free space

$$v = c = \frac{1}{(\mu_0\nu_0)^{1/2}} = 3 * 10^8$$

Theoretically the wavelength λ of these radiations may have values from $\lambda \to 0$ to $\lambda \to \infty$. Some examples of em waves are- visible light ($4000\mathring{A}$ *to* $7800\mathring{A}$) ,X-rays ($0.1\mathring{A}$ *to* $10\mathring{A}$) etc.

2.2 Reaction of matter on radiation

Let us see what happens when some radiation is incident on a surface? There are three possibilities some part of radiation may be reflected some part may be absorbed and some may be transmitted. From conservation of energy we can write

$$Q = Q_a + Q_r + Q_t \tag{2}$$

where

$$Q = \texttt{total energy of incident radiation}.$$

$$Q_a = \texttt{energy absorbed}$$
$$Q_r = \texttt{energy reflected}$$
$$Q_t = \texttt{energy transmitted}$$

or

$$1 = a + r + t \tag{3}$$

where

$$a = Q_a/Q = \texttt{absorption power}$$
$$r = Q_r/Q = \texttt{reflection power}$$
$$t = Q_t/Q = \texttt{transmission power}$$

2.3 Blackbody

A body which absorbs all the incident radiation is called blackbody. Clearly for blackbody $a = 1, r = t = 0$ The best black bodies in nature are lamp black ($a = 0.96$) and pt black ($a = 0.98$). But for experimental purposes one can prepare a cavity having $a \approx 1$ with inner walls polished by pt black.

2.4 Radiation by Hot Bodies

When a body is heated up it changes its color. That is because it emits radiations. At first we see that the color of hot body is red then it becomes some orange and at last after reaching a certain temperature it becomes white. This all signifies that hot body emits radiation of high λ at low temperature and at high temperature it also emits the radiation of lower λ .

2.5 Blackbody Radiation

When a black body is heated up and kept at a fixed temperature then em radiations are emitted from inner walls of cavity (blackbody). This radiation is partially reflected and absorbed by the inner walls of the blackbody itself. After some time the cavity is filled up by em radiation. The temperature of the blackbody is kept fixed this implies that the radiation is in thermal equilibrium. This means that em radiation emitted per second by the walls of cavity is the same as absorbed . In thermal equilibrium the energy density of the em radiation remains constant. as we have a hole in the cavity the radiation comes out from this hole which is a part of the inside radiation.

2.6 Energy Density

At a particular temperature T energy in the cavity per unit volume is called energy density. It is denoted by u .

2.7 Spectral energy Density

At a certain temperature the average energy density between wavelength λ and $\lambda + d\lambda$ is given by $u_\lambda d\lambda$. where $u - \lambda$ is called spectral energy density. the relation between u and u_λ is

$$U = \int_0^\infty u_\lambda d\lambda \tag{4}$$

One can use u_ν instead of u_λ . Which is defined between frequency range ν and $\nu + d\nu$.

2.8 Emissive Power

Total radiation energy emitted by unit surface area of the blackbody at a certain temperature is called Emmisive power of the blackbody. It is denoted by E.

2.9 Spectral Emissive Power

Total radiated energy between λ and $\lambda + d\lambda$ range emitted by unit surface area of blackbody at certain temperature is given by

$$E_\lambda d\lambda$$

Where E_λ is called spectral emissive power of blackbody. It is clear from the definition that

$$E = \int_0^\infty E_\lambda d\lambda \tag{5}$$

If we know E_λ at a certain temperature we can find u_λ by the following relation

$$U = \frac{4}{c} E_\lambda \tag{6}$$

Where c is the speed of light.

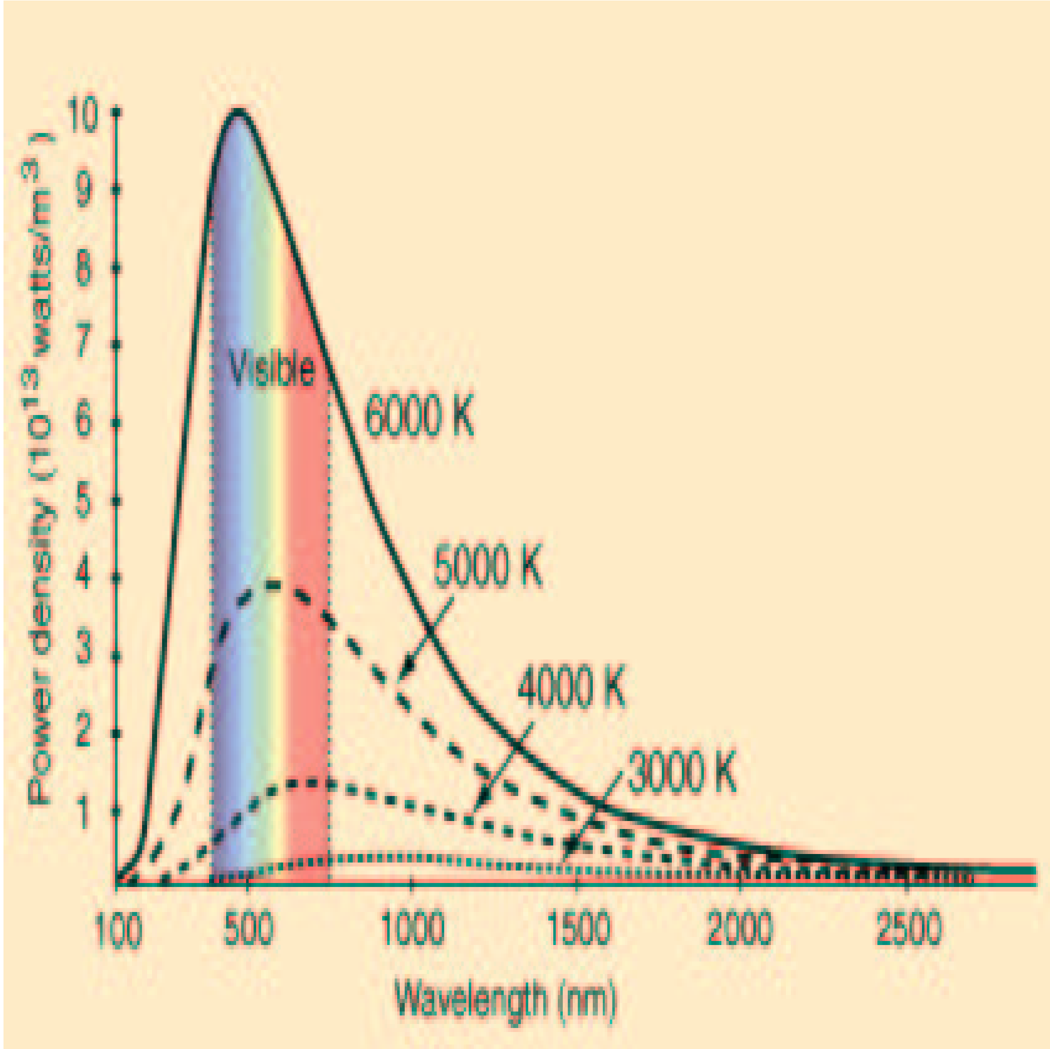


Figure 1: figure shows the experimental curve of blackbody radiation

3 Experimental study of Blackbody Radiation

First attempt was made by Lummer and Pringsheim in 1899. They plotted some curves between E_λ and λ for various temperature as shown infig(1).These plots are known as spectral energy distribution curve of blackbody radiation.

3.1 Characteristics of Blackbody Radiation

1. As it is clear from the figure the graph is continuous which means that at every temperature radiation for all wavelengths emitted but the spectral emissive power is different for different wavelength.

2. Spectral energy density $E\lambda$ for each λ increases with temperature.

3. At a particular temperature at first $E\lambda$ increases with λ but after reaching a

certain highest value it goes on decreasing. That highest value is denoted by E_{λ_m} and the wavelength at which E_λ is maximum is denoted by λ_m

4. Wien's Displacement Law-

   As we see from the graph λ_m (corresponding wavelength for maximum emission) decreases with temperature. It was Wien who first discovered mathematically that

$$\lambda_m \propto \frac{1}{T}$$

   or

$$\lambda_m = \frac{b}{T} \tag{7}$$

   Where b is called Wien's constant its value is $b = 2.898 * 10^{-3} meter Kelvin$ The above law is known as Wien's displacement law. This is very important law as it law helps us to find the temperature of stars(hot bodies).

   Wien's displacement law may be expressed in terms of frequency as

$$\nu_m = \frac{cT}{b} \tag{8}$$

5. We also see that the peak of graph increases rapidly with temperature. It is found that

$$E\lambda_m \propto T^5$$

6. Stephan Boltzmann's Law- At a particular temperature the area under the curve is given by

$$\int_0^\infty E_\lambda d\lambda$$

   Which is the total emissive power of blackbody. Hence the area of the curve represents the total emissive power. It is found to be proportional $to T^4$ i.e.

$$E \propto T^4$$

   or

$$E = \sigma T^4 \tag{9}$$

   Where σ is known as Stephan's constant having value

$$\sigma = 5.67 * 10^{-8}\ \texttt{watt}/m^2/K^4$$

   This law is known as Stephan Boltzmann's law.

4 Failure of Classical Physics

Many attempts were made to explain the experimental curve of blackbody radiation which are discussed in the previous section. Now we will study Wien's and Rayleigh Jeans distribution law to explain the blackbody radiation. We will also see how these laws are unable to explain it completely.

5 Wien's Distribution Law

5.1 Distribution Law

William Wien used thermodynamics to show that the spectral energy density between λ and $\lambda + d\lambda$ range is given by

$$E_\lambda d\lambda = \frac{A}{\lambda^5} f(\lambda T) d\lambda \tag{10}$$

To find the form of function $f(\lambda T)$ he compared the blackbody radiation curve with the Maxwellian energy distribution curve. After the comparison he deduced

$$f(\lambda T) = e^{-a/\lambda T}$$

And hence he found that

$$E_\lambda d\lambda = A\lambda^{-5} e^{-a/\lambda T} d\lambda \tag{11}$$

Where A and a are some constants. The above equation is known as Wien's Distribution Law.

5.2 Wien's Explanation

For the short values of λ exponential factor becomes large and contributes more which overcomes the other factor λ^{-5} . This means that at shorter wavelengths E_λ increases with λ . On the other hand at higher λ exponential factor is very small. In this range λ^{-5} factor dominate mostly and hence E_λ should be decreased at higher λ .

In the first sight we find the Wien's law good to to explain blackbody radiation curve. But compare the curve plotted by Wien's distribution law with the experimental one. As we see that in the shorter λ range Wien's law fits very well but we find a difference between these curves in the higher λ range. This implies an error in the theoretical distribution law which is too large to ascribe to experimental uncertainties and indicates a flaw in the theory. Wien could neither explain the failure of his relation nor supply a better one.

Although Wiens law does not hold good for complete explanation but one can deduce the maximum spectral emissive power dependence on temperature by this as follows-

From Wiens displacement law we have at $\lambda = \lambda_m, \;\; \lambda_m T = b$.

Using this in Wiens distribution law

$$E\lambda_m = A{\lambda_m}^{-5} e^{-a/\lambda_m T}$$

$$= A{\lambda_m}^{-5} e^{-a/b}$$

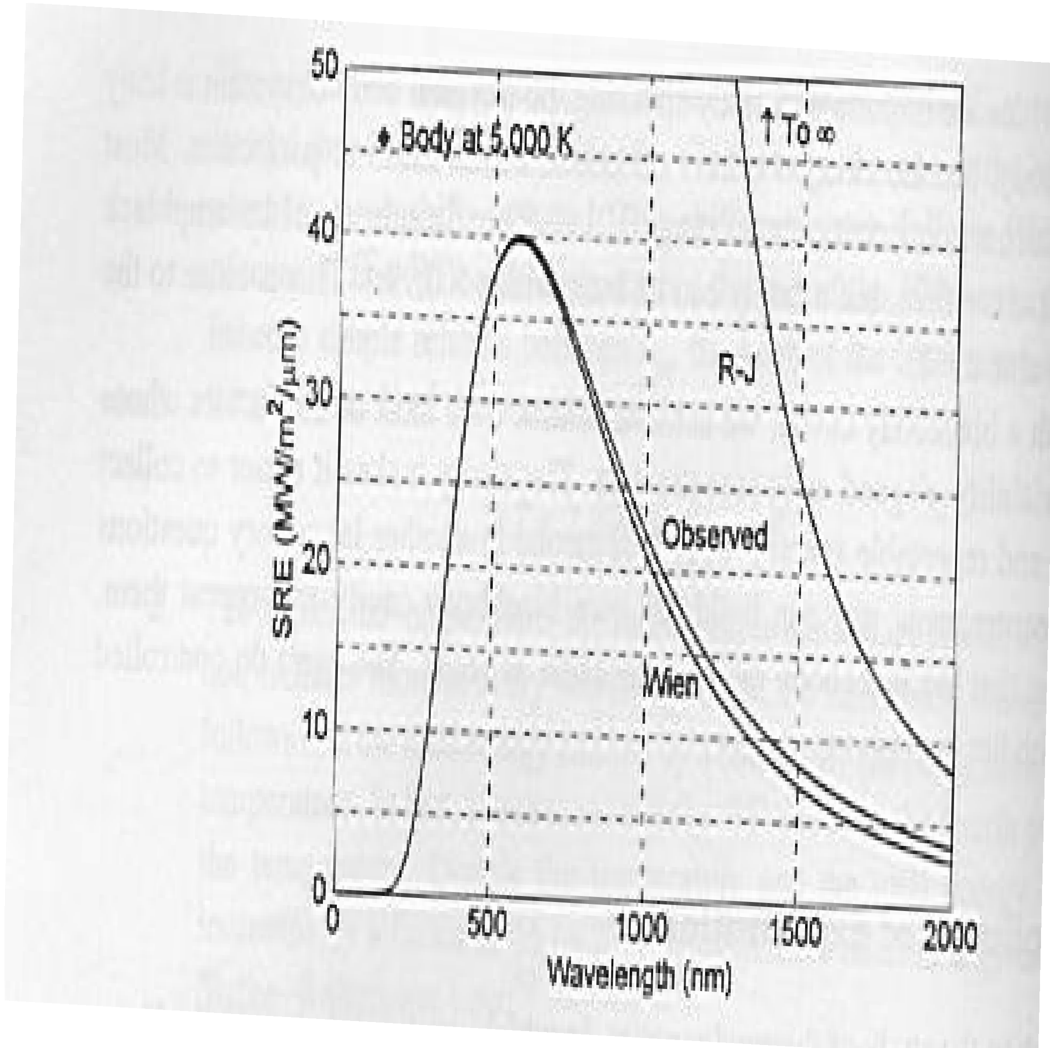


Figure 2: this figure shows the plot of wien's distribution law

$$\Rightarrow E\lambda_m \propto {\lambda_m}^{-5}$$

$$\Rightarrow E\lambda_m \propto T^5$$

Similar to that found in experimental curve.

6 Rayleigh Jean's Distribution Law

In 1900 Rayleigh and Jeans used Maxwell Boltzmann statistics to derive another distribution in order to explain blackbody radiation.

6.1 Calculation of modes

According to Rayleigh the radiation waves in blackbody can be compared to the standing waves in cubical cavity. Let the equation of standing wave in cubical cavity is

$$y = Asin(\mathbf{k} \cdot r)cos\omega t$$

or

$$y = Asin(k_x + k_y + k_z)cos\omega t \tag{12}$$

For standing wave at the wall their must be only nodes as shown in the figure. In order to satisfy this condition

$$k_x = \frac{n_x \pi}{a}$$

$$k_y = \frac{n_y \pi}{a}$$

$$k_z = \frac{n_z \pi}{a}$$

Where n_x, n_y, n_z are $1, 2, 3 \cdots N$ and a is the length of the cubical cavity.

Now

$$k^2 = {k_x}^2 + {k_y}^2 + {k_z}^2 = \frac{{n_x}^2 + {n_y}^2 + {n_z}^2}{\pi^2 a^2}$$

or

$${n_x}^2 + {n_y}^2 + {n_z}^2 = \frac{k^2 a^2}{\pi^2}$$

but

$$k = \frac{2\pi}{\lambda} = \frac{2\pi\nu}{c}$$

since radiation propagates with speed c

$${n_x}^2 + {n_y}^2 + {n_z}^2 = \frac{4a^2\nu^2}{c^2} \tag{13}$$

Which represents the equation of an sphere of radius $\frac{2a\nu}{c}$. Positive value values of n_x, n_y, n_z lies in the $8th$ part of this sphere.

And we know that one set of $n_x n_y n_z$ represents one mode of vibration. This implies that the total number of modes in frequency range $0 - \nu$ will be equal to the points in the 8th part of the volume of the sphere

$$= \frac{1}{8}\frac{4\pi}{3}\left(\frac{2a\nu}{c}\right)$$

$$= \frac{4\pi}{3}(\frac{a^3\nu^3}{c^3})$$

taking volume of the cavity as unity i.e. $a^3 = 1$ number of modes of vibrations in range $0 - \nu$

$$= \frac{4\pi}{3}\frac{\nu^3}{c^3}$$

The number of modes between range ν and $\nu + d\nu$

$$= \frac{4\pi\nu^2}{c^3}d\nu$$

Again we know that there are two polarization states of vibration for em waves hence per unit volume number of modes of vibrations

$$N_\nu d\nu = \frac{8\pi\nu^2}{c^3}d\nu \tag{14}$$

Similarly if one wants to calculate number of modes in range λ to $\lambda + d\lambda$ he will find

$$N_\lambda d\lambda = \frac{8\pi}{\lambda^4}d\lambda \tag{15}$$

6.2 complete distribution law

In an isothermal enclosure, the radiation is constant both in quantity and in spectral characteristics, so that any energy absorbed is reradiated. The result is the same as if all the energy were reflected at the walls. Rayleigh and jeans assumed that oscillators in the wall absorbed and emitted radiation constantly, with each oscillator having its own characteristic frequency. for continuous operation of any given oscillation, standing waves must be set up in the enclosure. however for any enclosure of reasonable size the differences between neighboring frequencies are so small that the radiation appears to be continuous.

The principle of equipartition of energy requires assigning to these oscillators $1/2kT$ of kinetic energy per degree of freedom plus another $1/2kT$ for potential energy. assignment of an average energy of kT to each mode of vibration leads to an energy density $U_\lambda d\lambda$ for waves with wavelength between λ to $\lambda + d\lambda$ given by

$$U_\lambda d\lambda = \frac{8\pi kT}{\lambda^4}d\lambda \tag{16}$$

Above equation as the Rayleigh Jeans formula for blackbody radiation. It should be noted that it contains no new constants.

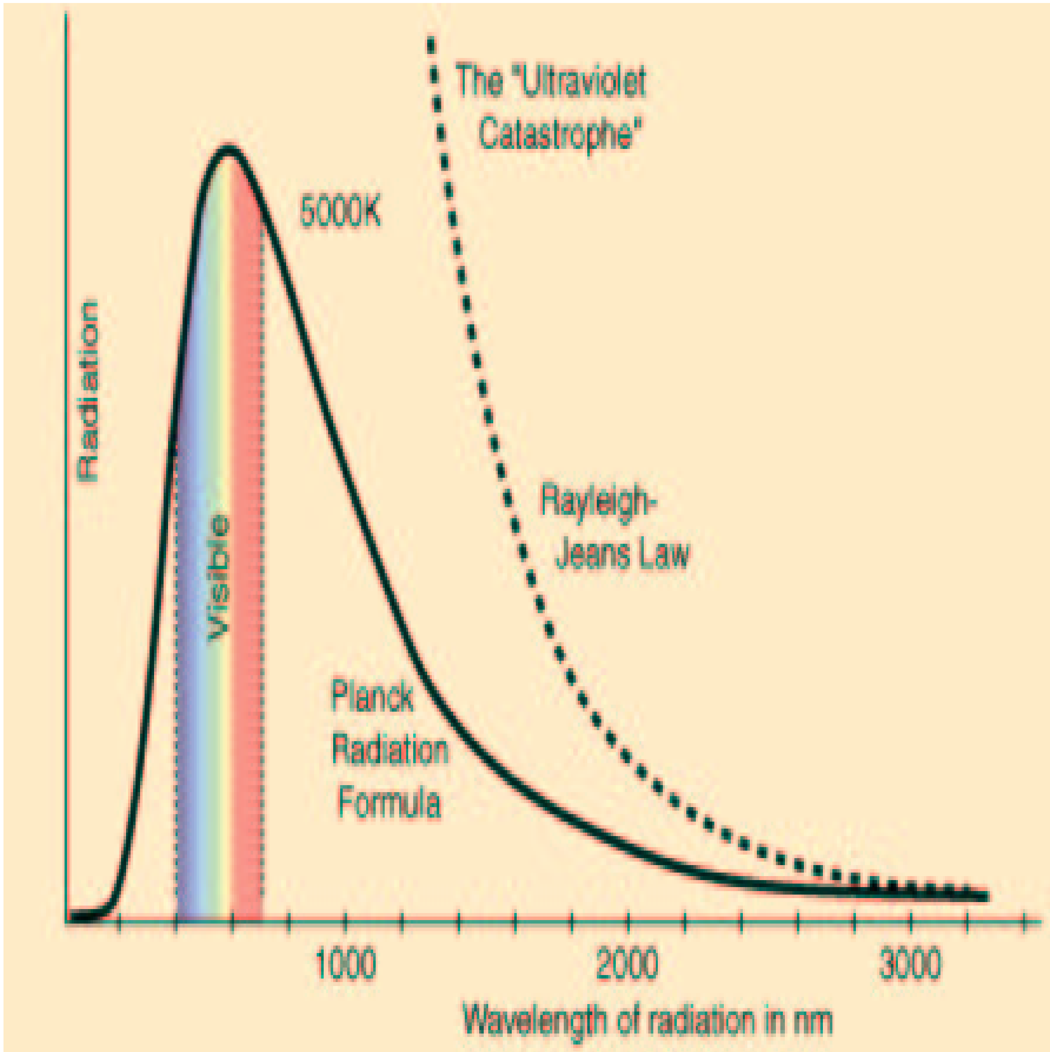


Figure 3: This figure shows ultravoilet catastrophe in classical physics.

6.3 Explanation by Rayleigh Jeans Law

From equation we see that energy density decreases with λ . Which is correct only in the higher wavelength region. Hence Rayleigh Jeans distribution law holds only for larger λ part. At small λ this law fails.

The most disturbing aspect of Rayleigh Jeans law is that the area under the curve is infinite. This area represents the total energy radiated by the blackbody. The Rayleigh jeans law forecasts that a body at a temperature let 500K will radiate energy at an infinite rate! there is nothing about 500k the law predicts infinite total radiation for all blackbodies, even those with temperature0.01K . This nonsensical result became known as the ultraviolet catastrophe because the excess radiation is found at short wavelength in the UV region. it signaled a fundamental flaw in classical thermodynamics. As was the case with Wilhelm Wien , Lord Rayleigh could not improve the prediction.

6.4 Failure of Classical Physics

Thus we have two theoretical relations equation 11 and 16 produced by first rate thermodynamicists to explain blackbody radiation. One is good at short wavelength but inadequate at long;the other is poor in the IR and ridiculous in the UV . That two outstanding physicists could not produce satisfactory equations for the radiation of a blackbody boded ill for the future of classical thermodynamics.

7 New Idea

As we have already seen that classical laws of physics were unable to solve the Blackbody Radiation puzzle. In order to solve it Planck gave a completely new and brave idea. Which is known as quantum hypothesis. In this section we will see what the quantum theory is and how it helps to solve the problem of blackbody radiation.

7.1 Equipartition

If energy is made available to an isolated system and is absorbed by the molecules (or atoms) of the system, the question may be asked: In what manner will the molecules incorporate the energy? There are different modes in which an excited molecule can act. Equipartition held that the energy will divide equally among the various modes.

Consider, for example, a box containing a monatomic gas (that is, each molecule consists of a single atom). In classical physics, you could think of each atom as a minute sphere, dashing about in the box, striking other and the boundary walls. The kinetic energy of an atom is exhibited in three modes: movement in the $\pm x$ direction, in the $\pm$ y direction, and in the $\pm$ z direction. No one has ever succeeded in adding energy to a gas in such a way that the kinetic energy of the atoms, measured along the x-axis, is different from their energy measured along either of the other axes.(On a time scale short compared with the mean interval between collisions in the gas, there might be momentary statistical fluctuations between the three components,but over longer periods of time the three components of energy would be equal.)Invariably, if you add X joules of energy to the box, $X/3$ joules will appear as increased kinetic energy of the atoms in each of the three coordinate directions. You cannot preferentially augment the kinetic energy in some special direction. This is an example of equipartition.

If, instead of working with a monatomic gas, you worked with a gas whose components were complicated molecules(that is, more than one atom forming each molecule), new modes of energy are possible. a molecule, in response to absorbing energy, could speed up(as did the monatomic gas), but it could also spin up or develop internal vibrations. There would be more than three modes in which the added energy could manifest itself. Again, equipartition held that the energy added would be equally divided among all possible modes.

A similar situation was thought to apply in a blackbody cavity as well. The cavity is filled with radiation in the form of EM waves of different wavelengths. Each wavelength is associated with a mode, and the energy in the cavity is allocated among the modes. But(unlike the case of energetic atoms) the number of possible modes is infinite. This is so because radiation in the cavity was thought to appear in the form of sinusoids whose wavelengths permit them to exist as standing waves.A sinusoid stretching between opposite walls of the cavity must have nodes where it touches those walls. To meet this requirement, a half wave of a sinusoid must exactly fit between the walls, or two half waves must fit, or three, etc. In other words, a sinusoid can form a standing wave in the cavity if the cavity width, a, is a multiple of the half wavelength, $\frac{1}{2}$. The ratio of these two, must be an integer. This means that a standing wave must have a wavelength of $2a$,or a, or $2a/3$, or $2a/4$, etc., because a sinusoid with one of these wavelengths will have nodes at the walls. There are an infinite number of permitted sinusoids, or modes.

According to the venerable principle of equipartition, when energy is added to a cavity it must divide equally among the allowed modes. thus, if any allowed mode is energized, all such modes receive equal energy. The Ultraviolet Catastrophe immediately follows because there are an infinite number of allowed modes, almost all of them at short wavelengths. Of course, there is something wrong here: if 1 W of energy is added to the cavity, each of the allowed modes receives an infinitesimally small amount of energy. What does that mean? Conversely, of each mode is energized with as little as a pico- pico- pico- picowatt, the total radiation is infinitely large. Both possibilities make our minds reel.

Planck didn't believe that equipartition was wrong: he thought it was probably correct. but he had convinced himself that all the remaining steps in the derivation of the spectral equation were valid. He was less sure about equipartition: it was the weak link in the chain. In an act of desperation, he invented an alternative assumption. He later wrote:

It was an act of desperation. For six years i had struggled with the blackbody theory. I knew the problem $\cdots$ and i knew the answer. I had to find a theoretical explanation at any cost

While assuming, as had all his predecessors, that the walls of the cavity consisted of an indefinitely large number of oscillators which could gain or lose energy, and so determine the energy in the interior of the cavity, he postulated, as no one had before him, that the energy of the oscillators was quantized.

7.2 What is Quantized Oscillator?

Planck's new and radical assumption was that an oscillator in the cavity walls could emit or absorb energy only in chunks (or quanta, to use the proper term and that the chunk size was proportional to the frequency of the oscillator. If the oscillator operates at a frequency ν, the size of its quantum of energy is $h\nu$here, h is a small constant(whose value Planck did not at first know) with the units of energy $\cdot$ time : a quantity with the name action. If this oscillator ν is to accept energy from the cavity, it can accept only one quantum's worth, the amount $h\nu$.The oscillator can not, under Planck's hypothesis, gain or lose energy except in this precise amount. Multitudes of other oscillators are located in the cavity walls, Planck argued, each with its own frequency, and the full set of oscillators covers all possible frequencies. Yet each oscillator can accept or release only the quantum $h\nu$ of energy, where ν is its particular frequency.

Planck's assumption of quantized oscillators had a major effect on the distribution of radiant energy in the cavity. We'll illustrate this by an example using small numbers: suppose that the fundamental frequency in the cavity (the lowest frequency that any oscillator could work at in that size cavity, as determined by the size, a, of the cavity) is ν , and that during some brief interval of time $6h\nu$ of energy becomes available from the heating coils. This energy can excite oscillators in the walls of the cavity in a variety of ways, among which the following are representative:

- Six oscillators at the fundamental frequency ν could each absorb 1 quantum of energy, $h\nu$. Each would then be energized in the fundamental mode and supply this energy to the cavity radiation. Note that we have accounted for all the energy : six oscillators each gained $h\nu$ of energy. in equation form,

$$6 * 1h\nu = 6h\nu$$

- Or, four oscillators at the fundamental might each absorb one quantum (that is, $1h\nu$) while a fifth oscillator at the second harmonic (2ν) absorbs the remaining $2h\nu$ of energy. the energy available to the cavity takes the form of 4 quanta at frequency ν and 1 quantum at frequency 2ν . The accounting equation is

$$4 * 1h\nu + 1 * 2h\nu 6h\nu$$

- Or, one oscillator might be excited at the fundamental, one oscillator at the second harmonic, and one oscillator at frequency 3ν . Then the cavity receives $h\nu$ of energy at the fundamental, $2h\nu$ at frequency 2ν and $3h\nu$ at frequency 3ν. the accounting relation is

$$1 * 1h\nu + 1 * 2h\nu + 1 * 3h\nu = 6h\nu$$

- ...

- Or, to give a final (but not exhaustive) example, one oscillator might absorb all 6 quanta and become excited at frequency 6ν . The cavity would be presented with $6h\nu$ of energy at the sixth harmonic. The accounting relation would be

$$6 * 1h\nu = 6h\nu$$

We see that the available quanta can be allocated to oscillators in many ways (of which the foregoing are but a few). We have illustrated the arithmetic of how the quanta and the oscillators can interact. there are two rules, an old one and a new one :

1. The total energy supplied by the heating coils must equal the amount absorbed by various oscillators, and also equal the total amount released by those oscillators into the cavity.

2. An oscillator at frequency ν can accept or release only the quantum, $h\nu$ of energy.

We realize that, under Planck's logic - an alternative, remember, to equipartition - the higher frequencies are discriminated against. In our trivial example, the available energy cannot excite any oscillator at a frequency of 7ν or higher. To do so would require at least $7h\nu$ of energy, which is more than we have assumed available. Of course, if more than 6 quanta were delivered by the heating coils, frequencies higher than the sixth harmonic would be excited, but you can see how Planck's novel assumption prevents the high frequencies from taking over, as it were. as you recall , it was the high frequency modes (short wavelength, large ν), each demanding its equal share of energy, that gave rise to the UV catastrophe in classical thermodynamics.

8 Planck's Distribution Law

8.1 Average eEnergy of Planck Oscillator

Planck used Maxwell Boltzmann statistics to calculate radiation energy. According to Planck at temperature T the number of oscillators having energy $nh\nu$ is

$$N_n = Aexp(\frac{-nh\nu}{kT}) \tag{17}$$

Where k is Boltzmann constant.

The total energy of oscillators having energy $nh\nu$ is

$$E_n = nh\nu N_n \tag{18}$$

Hence the total energy of all oscillators in the blackbody

$$E = \Sigma E_n = \Sigma nh\nu N_n$$

Total number of oscillators in the blackbody is

$$N = \Sigma N_n$$

$\Rightarrow$ average energy of oscillator is

$$\langle E \rangle = \frac{\texttt{total energy of oscillator}}{\texttt{total number of oscillator}}$$

or

$$\langle E \rangle = \frac{\Sigma nh\nu N_n}{\Sigma N_n}$$

$$= \frac{\Sigma nh\nu exp(\frac{-nh\nu}{kT})}{\Sigma exp(\frac{-nh\nu}{kT})}$$

let

$$x = \frac{h\nu}{kT}$$

$$\Rightarrow \langle E \rangle = kT \frac{\Sigma nxe^{-nx}}{\Sigma e^{-nx}}$$

$$= -xkT \frac{\frac{d}{dx}[\Sigma e^{-nx}]}{\Sigma e^{-nx}}$$

but

$$\Sigma e^{-nx} = \frac{1}{1 - e^{-x}}$$

and

$$\frac{d}{dx}[\Sigma e^{-nx}] = \frac{-e^{-x}}{(1 - e^{-x})^2}$$

$$\Rightarrow \langle E \rangle = \frac{xkT}{(e^x - 1)}$$

Hence average energy of an oscillator is

$$\langle E \rangle = \frac{h\nu}{e^{h\nu/kT} - 1} \tag{19}$$

Or in terms of wave length

$$\langle E \rangle = \frac{hc/\lambda}{e^{hc/\lambda kT} - 1} \tag{20}$$

8.2 Complete Distribution Law

According to Planck energy density between range λ to $\lambda + d\lambda$ is

$$u_\lambda d\lambda = N_\lambda d\lambda \langle E \rangle$$

Planck used the calculation made by Rayleigh Jeans for number of oscillations. Hence

$$N_\lambda d\lambda = \frac{8\pi}{\lambda^4} d\lambda$$

thus

$$u_\lambda d\lambda = \frac{8\pi hcd\lambda}{\lambda^5 exp(\frac{hc}{\lambda kT}) - 1} \tag{21}$$

In terms of frequency

$$U_\nu d\nu = \frac{8\pi h\nu^3 d\nu}{c^3 exp(\frac{h\nu}{kT}) - 1} \tag{22}$$

The above equation is called Planck,s distribution law.

8.3 Let $h \to 0$

Having postulated quantization and derived a fine radiation law (it fitted the data spectacularly well and avoided the UV catastrophe), Planck let the quantization constant, k, slide toward zero as he had intended all along. Immediately he was back in trouble with his equations : the catastrophe reappeared and he found himself where his predecessors had failed. When h became zero, Planck's law, blurred into the classical laws that were so extravagantly wrong. Nothing he could do would keep matters satisfactory if h vanished. The constant had to remain finite or all that he had gained was lost.Furthermore, a unique value of the constant was indicated to give a best fit to experimental data. It bore in upon Planck- although he fought against the conclusion with all his might - that the quantization assumption was essential, not just a trick to manipulate the equations. Mathematically we see that as $h \to 0$

$$\langle E \rangle = kT$$

Which is the energy equipartition of energy.

9 Game of Planck's Distribution Law

Planck's distribution law can explain the problem of blackbody radiation with remarkable accuracy for all wavelengths. For small wavelengths it reduces to Wien's distribution law while in larger wavelength range it is similar to Rayleigh jeans law. We will see all these characteristics of blackbody radiation explained by Planck's law.

9.1 Explanation in small λ range

for $\frac{hc}{kT} \gg \lambda$

$$exp(\frac{hc}{kT}) \gg 1$$

using this in Planck's distribution law

$$u_\lambda d\lambda = \frac{8\pi hc}{\lambda^5}\frac{1}{e^{hc/\lambda kT}}d\lambda$$

$$= \frac{8\pi hc}{\lambda^5}e^{-hc/\lambda kT}d\lambda$$

This equation is similar to Wien's distribution law which is correct for small wavelengths. We have already discussed how can it explain the small λ region of blackbody radiation.

9.2 Explanation in large wavelength region

if $\frac{hc}{kT} \ll \lambda$ then

$$exp(\frac{hc}{kT}) \approx 1 + \frac{hc}{\lambda}$$

Using this in Planck's law we get

$$u_\lambda d\lambda = \frac{8\pi hc}{\lambda^5}\frac{1}{1 + hc/\lambda kT - 1}d\lambda$$

or

$$u_\lambda d\lambda = \frac{8\pi kT}{\lambda^4}d\lambda$$

Which is Rayleigh Jeans distribution law. It can explain all the properties of blackbody radiation at large wavelengths.

9.3 Calculation of Total Energy Density

Inside the blackbody total energy density is defined by

$$U = \int_0^\infty u_\lambda d\lambda$$

using plank distribution law

$$U = \int_0^\infty \frac{8\pi hc}{\lambda^5(e^{hc/\lambda kT} - 1)}d\lambda$$

let $\frac{hc}{\lambda kT} = x$ then differentiating it

$$d\lambda = \frac{-hc}{kTx^2}dx$$

using this

$$U = 8\pi hc \int_0^\infty \frac{-hc/kTx^2}{(hc/xkT)^5(e^x - 1)}dx$$

$$= \frac{8\pi k^4T^4}{c^3h^3}\int_0^\infty \frac{x^3}{e^x - 1}dx$$

but the standard integration

$$\int_0^\infty \frac{x^3}{e^x - 1}dx = \frac{\pi^4}{15}$$

$$\Rightarrow U = \left[\frac{8\pi^5k^4}{15h^3c^3}\right]T^4 \tag{23}$$

This equation represents the total energy density of blackbody.

9.4 Deduction of Stefan Boltzmann's law

The emissive power i.e. the energy radiated per second by unit surface area of the blackbody is

$$E = \frac{cU}{4}$$

$$= \frac{2\pi^5k^4}{15h^3c^2}T^4$$

or

$$E = \sigma T^4$$

Which is Stefan's law. Where

$$\sigma = \frac{2\pi^5k^4}{15h^3c^2} \tag{24}$$

9.5 Deduction of Wien's Displacement Law

Planck's distribution law is

$$u_\lambda = \frac{8\pi hc}{\lambda^5(e^{hc/\lambda kT} - 1)}$$

u_λ is maximum at $\lambda = \lambda_m$ then

$$\left[\frac{du_\lambda}{d\lambda}\right]_{\lambda_m} = 0$$

This gives

$$\frac{-5}{\lambda^6}\frac{1}{e^{hc/\lambda kT} - 1} - \frac{\lambda^2 kT}{\lambda^5(e^{hc/\lambda kT} - 1)^2} = 0$$

at $\lambda = \lambda_m$

or

$$5 = \frac{ch}{\lambda kT}\frac{e^{hc/\lambda kT}}{e^{hc/\lambda kT} - 1}$$

at $\lambda = \lambda_m$

let $x = \frac{hc}{\lambda kT}$ then above equation reduces to

$$e^x = \frac{5}{5-x}$$

or

$$x = ln5 - ln(5-x)$$

This is a non algebric equation having solution

$$x \approx 4.965$$

Hence $\frac{hc}{\lambda_m kT} = 4.965$ or

$$\lambda_m T = \frac{hc}{k(4.965)}$$

Substituting the values of $h, c\&k$

$$\lambda_m T = 2.989 * 10^{-3}$$

Which is Wien's displacement law.

10 Further Aspects of Blackbody Radiation

10.1 Kirchhoff's law for radiation

Kirchhoff's law is very important to understand the absorption and emission of radiation by hot bodies. It states that at a fixed temperature the emissive power of a body of a body is proportional to its absorption power.Mathematically

$$a_\lambda \propto e_\lambda \tag{25}$$

Kirchhoff's law helps us to understand the presence of Fraunhoffer lines in the spectrum of sun.

10.2 Thermodynamics of blackbody radiation

Blackbody radiation , like any system in the thermodynamic equilibrium , can be treated by thermodynamic methods . Let us make a blackbody enclosure with a piston, so that work may be done on or extracted from the radiation. now by the first law of thermodynamics, we have

$$dQ = dU + pdV \tag{26}$$

Where Q is heat and U is total energy. By the second law of thermodynamics,

$$dS = \frac{dQ}{T} \tag{27}$$

where $S = entropy$. $U = uV$ and $p = \frac{u}{3}$ from differentiation we have (using above two equations)

$$dS = \frac{V}{T}\frac{du}{dT}dT + \frac{u}{T}dV + \frac{1}{3}\frac{u}{T}dV$$

or

$$dS = \frac{V}{T}\frac{du}{dT}dT + \frac{4u}{3T}dV \tag{28}$$

since dS is a perfect differential,

$$\left[\frac{\partial S}{\partial T}\right]_V = \frac{V}{T}\frac{du}{dT} \tag{29}$$

$$\left[\frac{\partial S}{\partial V}\right]_T = \frac{4u}{3T} \tag{30}$$

using these equations one can easily get

$$u(T) = aT^4 \tag{31}$$

Which is an another form of Stephan's law.

By simple algebra we can also get

$$S = \frac{4}{3} a T^3 V \tag{32}$$

So that for adiabatic process

$$pV^{4/3} = \texttt{constant} \tag{33}$$

Which is adiabetic law as

$$pV^{\gamma} = \texttt{constant} \tag{34}$$

where $\gamma = 4/3$.

11 Applications of blackbody radiation

In astrophysics we generally deal with hot bodies like stars . We can use the physics of blackbody radiation to study such objects. For the purpose we define some specific temperatures as follows

11.1 Brightness Temperature

Consider a hot body (not blackbody) emitting radiation characterized by some brightness. Now consider a blackbody having same brightness. Then the temperature of this blackbody is called brightness temperature for the hot body. It is denoted by T_b . Mathematically

$$I_\nu = B_\nu(T_b) \tag{35}$$

Where

$$I_\nu = \texttt{specific intensity}$$

and

$$B_\nu = \texttt{Planck function}$$

This way of specifying brightness has the advantage of being closely connected with the physical properties of the emitter. This procedure is used especially in radio astronomy where the Rayleigh Jeans law is usually applicable. So that

$$I_\nu = \frac{2\nu^2 k T_b}{c^2}$$

$$\Rightarrow T_b = \frac{c^2 I_\nu}{2\nu^3 K} \tag{36}$$

for $h\nu \ll kT$

11.2 Color Temperature

Often a spectrum is measured to have a shape more or less of blackbody form, but not necessarily of the proper absolute value. For example , by measuring F_ν from an unresolved source we cannot find I_ν unless we know the distance to the source and its physical size. By fitting the data to a blackbody curve without regard to vertical scale, a color temperature T_c is obtained. Often the "fitting" procedure is nothing more than estimating the peak of the spectrum and applying Wien's displacement law to find a temperature. The color temperature T_c will correctly give the temperature of a blackbody source of unknown absolute scale. Also, T_c will give the temperature of a thermal emitter that is optically thin, providing that the optical thickness is fairly constant for frequencies near the peak. In this case the brightness temperature will be less than the temperature of the emitter, since the blackbody spectrum gives the maximum attainable intensity of a thermal emitter at temperature T, by general thermodynamic arguments.

11.3 Effective Temperature

The effective temperature of a source T_{eff} is derived from the total amount of flux, integrated over all frequencies, radiated at the source. We obtain T_{eff} by equating the actual flux F to the flux of a blackbody at temperature T_{eff} :

$$F = \int cos\theta I_\nu d\nu d\Omega \equiv \sigma T^4 \tag{37}$$

Note that both T_{eff} and T_b depends on the source intensity, but T_c depends only on the shape of the observed spectrum.

References

[1] Alan A. Grometstein *1999, The Roots of Things*

[2] Richtmayer Kernard Cooper *1969,Introduction To Modern Physics*

[3] Rybicki Lightman *1979,Radiative Processes In Astrophysics*

Heat capacities of solids

Any theory used to calculate lattice vibration heat capacities of crystalline solids must explain two things:

1. Near room temperature, the heat capacity of most solids is around 3k per atom (the molar heat capacity for a solid consisting of n-atom molecules is ~3nR). This is the well-known Dulong and Petit law.

2. At low temperatures, C_v decreases, becoming zero at T=0. Heat capacities have a temperature dependence of the form $\alpha T^3 + \gamma T$, where the T^3 term arises from lattice vibrations, and the linear term from conduction electrons.

Classical mechanics would predict $C_v = 3R$ at all temperatures, in violation of both experiment and the third law of thermodynamics.

<u>Einstein's theory of heat capacities</u>

Einstein treated the atoms in a crystal as N simple harmonic oscillators, all having the same frequency ν_E. The frequency ν_E depends on the strength of the restoring force acting on the atom, i.e. the strength of the chemical bonds within the solid. Since the equation of motion for each atom decomposes into three independent equations for the x, y and z components of displacement, and N-atom solid is equivalent to 3N harmonic oscillators, each vibrating independently at frequency ν_E. Note that this treatment is a *gross* approximation, since in reality the lattice vibrations are very complicated coupled oscillations.

The energy levels of the harmonic oscillators are given by

$$\varepsilon_v = h\nu_E(v + \tfrac{1}{2}), \qquad v = 0, 1, 2\ldots$$

Assuming the oscillators are in thermal equilibrium at temperature T, the partition function for a single oscillator is

$$q = \sum_{v=0}^{\infty} \exp[-\beta\varepsilon_v] = \sum_{v=0}^{\infty} \exp[-\beta h\nu_E(v+\tfrac{1}{2})] = e^{-x/2}\sum_{v=0}^{\infty} e^{-xv} = \frac{e^{-x/2}}{1-e^{-x}} \quad \text{where} \quad x = \beta h\nu_E.$$

In the above, we have used the fact that $\sum_{n=0}^{\infty} x^n = \dfrac{1}{1-x}$.

The mean energy per oscillator is then

$$u = -\frac{d\ln q}{d\beta} = \frac{d}{d\beta}\left(\frac{\beta h\nu_E}{2} + \ln(1-e^{-\beta h\nu_E})\right) = \frac{h\nu_E}{2} + \frac{h\nu}{e^{\beta h\nu_E}-1}$$

The first term above, $h\nu/2$, is simply the zero point energy. Using the fact that energy is an extensive property, the energy of the 3N oscillators in the N-atom solid is

$$U = 3Nu = 3N\left(\frac{h\nu_E}{2} + \frac{h\nu}{e^{\beta h\nu_E}-1}\right)$$

The heat capacity at constant volume is therefore

$$C_v = \left(\frac{\partial U}{\partial T}\right)_v = 3N\left(\frac{\partial U}{\partial \beta}\right)_v\frac{\partial \beta}{\partial T} = 3Nk\,\frac{x^2e^x}{(e^x-1)^2} \quad \text{where} \quad x = \frac{h\nu_E}{kT} = \frac{\theta_E}{T}$$

θ_E is the 'Einstein temperature', which is different for each solid, and reflects the rigidity of the lattice.

At the high temperature limit, when $T \gg \theta_E$ (and $x \ll 1$), the Einstein heat capacity reduces to Cv = 3Nk, the Dulong and Petit law [prove by setting $e^x \sim 1+x$ in the denominator].

At the low temperature limit, when $T \ll \theta_E$ (and $x \gg 1$), $C_v \rightarrow 0$ as $T \rightarrow 0$, as required by the third law of thermodynamics. [Prove by setting $e^x-1 \sim e^x$ in the denominator for large x].

<u>Debye's theory of heat capacities</u>

Debye improved on Einstein's theory by treating the coupled vibrations of the solid in terms of 3N normal modes of vibration of the whole system, each with its own frequency. The lattice vibrations are therefore equivalent to 3N independent harmonic oscillators with these normal mode frequencies. For low frequency vibrations, defined as those for which the wavelength is much greater than the atomic spacing, $\lambda \gg a$, the crystal may be treated as a homogeneous elastic medium. The normal modes are the frequencies of the standing waves that are possible in the medium.

Debye derived an expression for the number of modes with frequency between ν and $\nu+d\nu$ in such a medium.

$$g(\nu)\,d\nu = \frac{4\pi V\nu^2}{v^3}\,d\nu = \alpha\,\nu^2 d\nu$$

where V is the crystal volume and v is the propagation velocity of the wave. As outlined above, this expression applies only to low frequency vibrations in a crystal. Debye used the approximation that it applied to all frequencies, and introduced a maximum frequency ν_D (the Debye frequency) such that there were 3N modes in total. i.e. $\int_0^{\nu_D} g(\nu)d\nu = 3N$. The Debye frequency corresponds to $\lambda = 2a$, when neighbouring atoms vibrate in antiphase with each other. With this approximation in place, Debye integrated over all of the frequencies to find the internal energy of the crystal, and then calculated the heat capacity using $C_v = \left(\frac{\partial U}{\partial T}\right)_v$. The resulting expression is given below.

$$C_v = 3Nk\left(\frac{3}{x_D{}^3}\int_0^{x_D}\frac{x^4\,e^x\,dx}{(e^x-1)^2}\right)$$

where $x = \frac{h\nu}{kT}$, and $x_D = \frac{h\nu_D}{kT} = \frac{\theta_D}{T}$. The Debye heat capacity depends only on the Debye temperature θ_D. The integral cannot be evaluated analytically, but the bracketed function is tabulated.

At high temperatures ($T \gg \theta_D$, $x_D \ll 1$), we may rewrite the integrand as follows:

$$\frac{x^4 e^x}{(e^x - 1)^2} = \frac{x^4}{(e^x - 1)(1 - e^x)} = \frac{x^4}{2(\cosh(x) - 1)} = \frac{x^4}{2(x^2/2! + x^4/4! + \ldots)}$$

Retaining only the x^2 term in the denominator gives

$$C_v = 3Nk\left(\frac{3}{x_D{}^3}\int_0^{x_D} x^2\, dx\right) = 3Nk$$

To determine the low temperature limit ($T \ll \theta_D$, $x_D \gg 1$), we note that the integrand tends towards zero rapidly for large x. This allows us to replace the upper limit by ∞ and turn the integral into a standard integral, to give

$$C_v = 3Nk\left(\frac{T}{\theta_D}\right)^3\left(3\int_0^\infty \frac{x^4 e^x}{(e^x - 1)^2}\right) = \frac{12}{5}\pi^4 Nk\left(\frac{T}{\theta_D}\right)^3$$

We see that the Debye heat capacity decreases as T^3 at low temperatures, in agreement with experimental observation. This is a marked improvement on Einstein's theory.

Free electron model of metals

Up to this point, we have only considered contributions to the heat capacity from vibrations within the solid. In metals, the free conduction electrons also contribute to the heat capacity. In the free electron model of metals, the conduction electrons are treated as a perfect gas obeying Fermi-Dirac statistics. Interactions of the electrons with the positively charged atomic ions and with the other electrons are neglected. This is not such a bad approximation as it may appear at first: the ions provide a positively charged background that partly screens the electrons from each other; and the residual collisions are often relatively unimportant - the energetically accessible final states are often already occupied, making any collisional excitation process forbidden by the Pauli exclusion principle.

The first step in deriving the heat capacity is to determine the density of states. We will first do this in momentum space, and then transform the result into an expression describing the density of states per unit energy.

The 3-dimensional Schrodinger equation for the translational motion of the electrons has the solutions

$$\phi_{n_1n_2n_3}(r) = A\sin\left(\frac{n_1\pi x}{L}\right)\sin\left(\frac{n_2\pi y}{L}\right)\sin\left(\frac{n_3\pi z}{L}\right)$$

with $k = \left(\frac{\pi n_1}{L}, \frac{\pi n_2}{L}, \frac{\pi n_3}{L}\right)$ and $k^2 = \frac{\pi^2}{L^2}(n_1{}^2 + n_2{}^2 + n_3{}^2)$.

The allowed values of k therefore form a cubic point lattice in k-space, with spacing π/L and volume per point $(\pi/L)^3$. Finding the number of normal modes of the standing wave wavefunctions with k between k and k+dk is equivalent to finding the number of lattice points between two spherical shells of radii k and k+dk in the positive octant of k-space. The number of k-vectors of magnitude $\leq$ k is

$$n_k = \frac{\text{Volume of region}}{\text{Volume per point}} = \frac{(1/8)\,(4/3)\pi k^3}{(\pi/L)^3} = \frac{Vk^3}{6\pi^2} \quad \text{where } V = L^3.$$

The number f(k) within an interval dk is found by differentiating this expression, giving

$$f(k)\,dk = \frac{dn_k}{dk}dk = \frac{Vk^2\,dk}{2\pi^2}$$

Since $k = 2\pi p/h$ (and therefore $dk = (2\pi/h)dp$), the density of states in momentum space is

$$f(p)\,dp = \frac{8\pi V p^2\,dp}{h^3}$$

where an extra factor of two has been added to account for the two possible spin states of the electrons. This expression may be converted to an energy density of states by substituting $\varepsilon = p^2/2m$ (and so $d\varepsilon = (p/m)dp$), to give

$$f(\varepsilon)\,d\varepsilon = \frac{4\pi V}{h^3}(2m)^{3/2}\,\varepsilon^{1/2}\,d\varepsilon$$

To determine the number of electrons with energies between ε and $\varepsilon+d\varepsilon$, we need to multiply the above expression, which gives the density of states at energy ε, with the probability $n(\varepsilon)$ of finding an electron in a given state with energy ε. Electrons are Fermions, and obey Fermi-Dirac statistics, so $n(\varepsilon)$ is given by the Fermi-Dirac distribution (Note: the Fermi-Dirac distribution is an analogue of the Boltzmann distribution for systems in which spin must be taken into account)

$$n(\varepsilon) = \frac{1}{\exp[\beta(\varepsilon-\mu)] + 1} \qquad \text{where } \mu \text{ is the chemical potential and } \beta=1/kT$$

The number of electrons with energy between ε and $\varepsilon+d\varepsilon$ is then

$$dN(\varepsilon) = n(\varepsilon)f(\varepsilon)d\varepsilon = \frac{1}{\exp[\beta(\varepsilon-\mu)]+1}\,\frac{4\pi V}{h^3}(2m)^{3/2}\,\varepsilon^{1/2}\,d\varepsilon$$

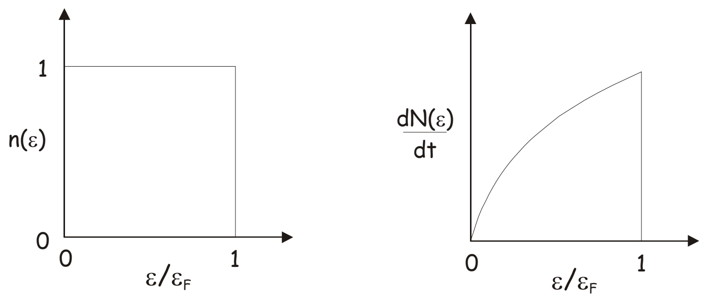
Integrating from zero to infinity gives the total number of electrons in the gas. The Fermi energy ε_F is the value of μ when T=0 i.e. $\varepsilon_F = \mu(0)$, and may also be written $\varepsilon_F = kT_F$, where T_F is the Fermi temperature.

Now we will look at the energy level occupations n(e) and the overall energy distribution $N(\varepsilon)$ as the temperature is increased from zero.

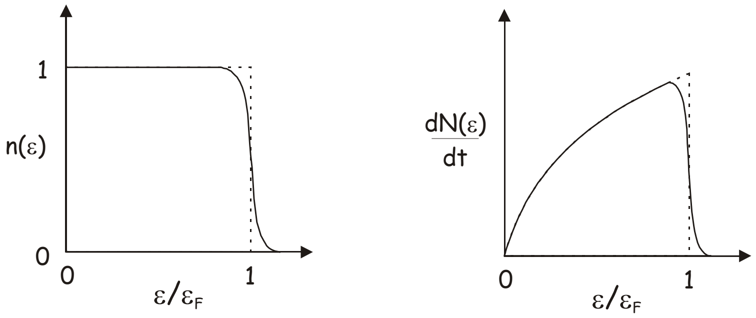
At T=0, the Fermi-Dirac distribution becomes

$$n(\varepsilon) = \frac{1}{\exp[\beta(\varepsilon-\varepsilon_F)] + 1}$$

and since $\beta=\infty$ at T=0, this is equal to 0 if $\varepsilon > \varepsilon_F$ and 1 if $\varepsilon < \varepsilon_F$. The two distributions $n(\varepsilon)$ and $N(\varepsilon)$ are shown below.



At higher temperatures, the two distributions change slightly from their behaviour at T=0, due to electrons lying below the Fermi level being excited to states lying above the Fermi level. This is shown below.



Now we will consider the consequences of these distributions for the heat capacity. Classical mechanics would predict a contribution ½ kT per electron to the heat capacity, in addition to the heat capacity arising from the lattice vibrations. This is a much larger contribution than is observed experimentally. The answer lies in the Fermionic nature of the electrons. In the classical model, a change in temperature ΔT would lead to an energy increase of around $k\Delta T$ per electron. However, we have seen that apart from electrons with energies very close to the Fermi energy ε_F, the states to which the electrons would be promoted by an energy increase of this magnitude are already occupied. Hence, only a very small fraction of electrons, those lying within ~$k\Delta T$ of the Fermi level, are able to absorb the energy and contribute to the heat capacity.

The heat capacity per electron turns out to be

$$C_V = \frac{\pi^2}{2} k \frac{T}{T_F}$$

At room temperature this is a very small contribution to the overall heat capacity (on the order of a few percent). However, at very low temperatures the electronic heat capacity dominates, since it is linear in temperature while the lattice heat capacity is proportional to T^3.